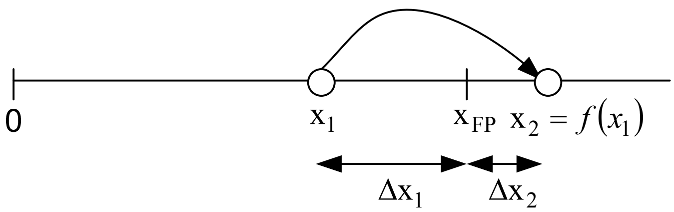
Calculation of the Jacobian of a point in an arbitrary orbit

1st order systems

Take as an example the logistic map: $x_{n+1} = f(x_n) = rx_n(1 - x_n) = rx_n - rx_n^2$

This has 2 FP: 0 and $\frac{r-1}{r}$, the Jacobian is $f'(x) = r - 2rx$

If we start close to the 2nd fixed point, i.e. we add a perturbation close to the FP:



$$x_2 = f(x_1) \Leftrightarrow x_{FP} + \Delta x_2 = f(x_{FP} + \Delta x_1) \approx f(x_{FP}) + f'(x_{FP})\Delta x_1 \Leftrightarrow$$
$$x_{FP} + \Delta x_2 = x_{FP} + f'(x_{FP})\Delta x_1 \Leftrightarrow \Delta x_2 = f'(x_{FP})\Delta x_1$$

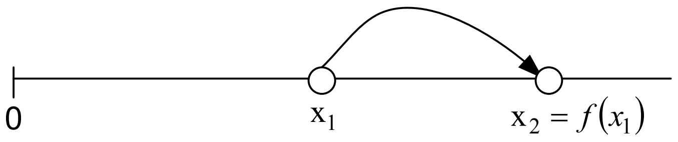
Thus if we want to find the Jacobian we: $f'(x_{FP}) = \frac{\Delta x_2}{\Delta x_1}$

Example:

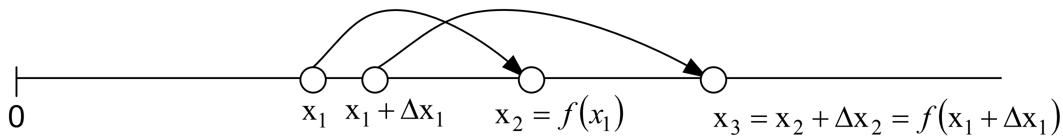
r=2.1, xep=(r-1)/r=0.52380952380952, x=1.001*(r-1)/r=0.52433333333333
Dx1=0.52433333333333-0.52380952380952=5.238095238100371e-004
x=r*x*(1-x)=0.52375656666667,
Dx2=0.52375656666667-0.52380952380952=-5.295714284991870e-005
J=-5.295714284991870e-005/ 5.238095238100371e-004=-0.10109999998611

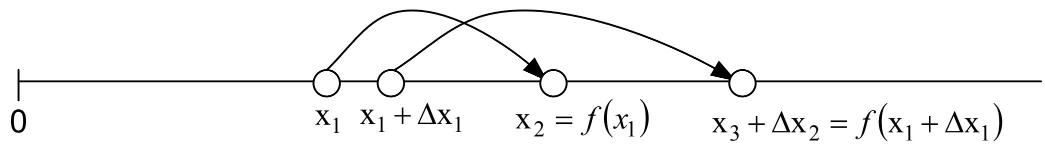
While from the formula J=r-2*r*0.52380952380952=-0.09999999999998

Now, let's try to see a generic point



Let's start close to x_1:





$$x_2 + \Delta x_2 = f(x_1 + \Delta x_1) \Leftrightarrow$$
$$x_2 + \Delta x_2 = f(x_1) + f'(x_1)\Delta x_1 \Leftrightarrow$$
$$x_2 + \Delta x_2 = x_2 + f'(x_1)\Delta x_1 \Leftrightarrow$$
$$\Delta x_2 = f'(x_1)\Delta x_1 \Rightarrow f'(x_1) = \frac{\Delta x_2}{\Delta x_1}$$

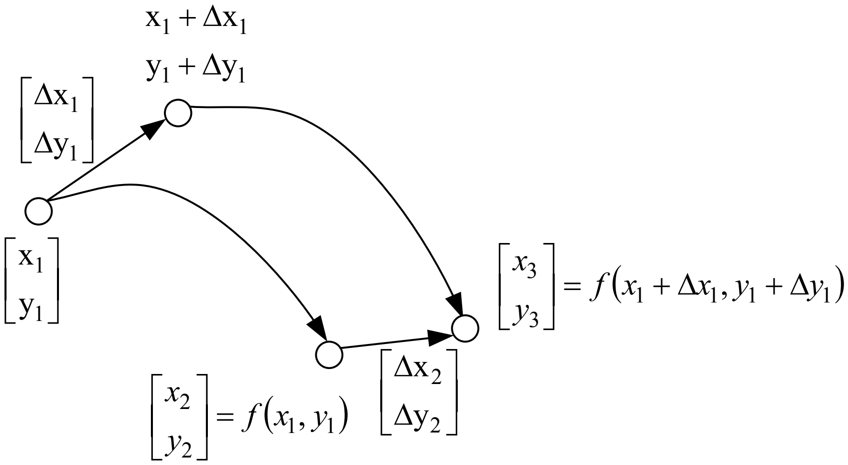
Example:
r=2.1, x=1.5, x=r*x*(1-x)=-1.575
x=1.5+0.001=1.501, x=r*x*(1-x)=-1.5792021
Dx1=0.001, Dx2=-1.5792021--1.575=-0.0042021
>> J=Dx2/Dx1=-4.20210000000010
From the formula: r-2*r*1.5=-4.2

2^{nd} order systems

Let's take the Henon map $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = f(x_n, y_n) = \begin{bmatrix} 1.8 - x_n^{\ 2} - 0.3y_n \\ x_n \end{bmatrix}$

The Jacobian is $J = \begin{bmatrix} -2x & -0.3 \\ 1 & 0 \end{bmatrix}$

Again our task is to find the Jacobian at an arbitrary point say $[x_1 \quad y_1]^T$



$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} \Delta x_2 \\ \Delta y_2 \end{bmatrix} = f(x_1 + \Delta x_1, y_1 + \Delta y_1) \Leftrightarrow$$
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} \Delta x_2 \\ \Delta y_2 \end{bmatrix} = f(x_1, y_1) + J(x_1, y_1)\begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} \Leftrightarrow$$
$$\begin{bmatrix} \Delta x_2 \\ \Delta y_2 \end{bmatrix} = J(x_1, y_1)\begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix}$$

Now, if want to find the Jacobian we need to repeat that so that we can have:

$$\begin{bmatrix} \Delta x_4 \\ \Delta y_4 \end{bmatrix} = J(x_1, y_1)\begin{bmatrix} \Delta x_3 \\ \Delta y_3 \end{bmatrix}$$

Then we can assume that we have $J(x_1, y_1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and then we have a 4 by 4 system that can be solved:

$$\left.\begin{array}{l}\begin{bmatrix}\Delta x_2\\ \Delta y_2\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta y_1\end{bmatrix}\\ \begin{bmatrix}\Delta x_4\\ \Delta y_4\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\Delta x_3\\ \Delta y_3\end{bmatrix}\end{array}\right\}\Rightarrow\left.\begin{array}{l}\Delta x_2=a\Delta x_1+b\Delta y_1\\ \Delta y_2=c\Delta x_1+d\Delta y_1\\ \Delta x_4=a\Delta x_3+b\Delta y_3\\ \Delta y_4=c\Delta x_3+d\Delta y_3\end{array}\right\}\Leftrightarrow$$

$$\begin{bmatrix}\Delta x_2\\ \Delta y_2\\ \Delta x_4\\ \Delta y_4\end{bmatrix}=\begin{bmatrix}\Delta x_1 & \Delta y_1 & 0 & 0\\ 0 & 0 & \Delta x_1 & \Delta y_1\\ \Delta x_3 & \Delta y_3 & 0 & 0\\ 0 & 0 & \Delta x_3 & \Delta y_3\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}\Leftrightarrow$$

$$\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=\begin{bmatrix}\Delta x_1 & \Delta y_1 & 0 & 0\\ 0 & 0 & \Delta x_1 & \Delta y_1\\ \Delta x_3 & \Delta y_3 & 0 & 0\\ 0 & 0 & \Delta x_3 & \Delta y_3\end{bmatrix}^{-1}\begin{bmatrix}\Delta x_2\\ \Delta y_2\\ \Delta x_4\\ \Delta y_4\end{bmatrix}$$

As an example lets take the point [0.6 0.6] which has a Jacobian $J=\begin{bmatrix}-1.2 & -0.3\\ 1 & 0\end{bmatrix}$

I add a perturbation Dx=[0.001;0.001] and I get the new point:

```
xa=0.6; ya=0.6; xa1=ro-xa^2+C1*ya; ya1=xa;

xb=0.6+0.001; yb=0.6+0.001; xb1=ro-xb^2+C1*yb; yb1=xb;

DX1=[xb-xa; yb-ya];

DX2=[xb1-xa1; yb1-ya1];

xc=0.6+0.002; yc=0.6-0.001

xc1=ro-xc^2+C1*yc; yc1=xc;

DX3=[xc-xa; yc-ya];

DX4=[xc1-xa1; yc1-ya1];

>> Dx1=DX1(1); Dy1=DX1(2);
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>> Dx2=DX2(1); Dy2=DX2(2);

>> Dx3=DX3(1); Dy3=DX3(2);

>> Dx4=DX4(1); Dy4=DX4(2);

>> J=[Dx1 Dy1 0 0;0 0 Dx1 Dy1; Dx3 Dy3 0 0; 0 0 Dx3 Dy3];

>> A1=inv(J)*[Dx2; Dy2; Dx4; Dy4]

  -1.201666666666618

  -0.299333333333355

   1.000000000000000

               0
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Which are the elements of my Jacobian!